ON SMALL PERTURBATIONS OF A PLANE PARALLEL FLOW WITH CUBIC VELOCITY PROFILE

(O MALYKH VOZMUSHCHENIIAKH PLOSKOPARALLEL'NOGO TECHENIIA S KUBICHESKIM PROFILEM SKOROSTI)

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R.V.BIRIKH
(Perm')

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The behavior of small perturbations in a plane-parallel antisymmetric steady flow with cubic velocity profile is investigated. Such a flow originates in a viscous fluid enclosed between vertical parallel planes heated to different temperatures [1]. The termal perturbations may be neglected in the limiting case of low Prandtl numbers, and the stability problem becomes purely hydrodynamic. The presence of an inflection point in the profile of the fundamental flow results in instability of the flow in the inviscid approximation [2]. The numerical solution of the Orr-Sommerfeld equation for the perturbations of the considered flow, whose numerical results ard presented herein, discloses a monotonous instability in the viscous flow, which sets in at comparatively low values of the Reynolds number R .

Results of a computation of the spectrum of the normal perturbations decrements in the $0 - 1500$ range of the numbers αR for various values of the wave number α and a neutral curve of the monotonous instability are presented herein. The stream functions for the monotonous perturbations which grows with time and for the damped vibrational perturbation are found for the values $\alpha = 1$ and $R = 1000$.

1. Let us consider the stability of the plane-parallel flow of a viscous incompressible fluid between the planes $x = \pm h$. Let

$$v_z = U_0\left[\frac{x}{h} - \left(\frac{x}{h}\right)^3\right]$$

be the velocity profile of the fundamental flow; z is the coordinate along the flow. Such motion occurs in a homogeneous fluid subject to a mass force dependent linearly on the x coordinate and independent of z . If U_0 , h and h^2/ν (ν is the kinematic viscosity) are taken as the units of measurement of the velocity, distance and time, respectively, and the stream function of a small normal perturbation is written as

$$\Psi(x, z, t) = \varphi(x)\exp(-\lambda t + i\alpha z)$$

(where α is the real wave number, λ the complex perturbation decrement), we then obtain the Orr-Sommerfeld equation for the perturbation amplitude $\varphi(x)$

$$\varphi^{IV} - 2\alpha^2\varphi'' + \alpha^4\varphi = i\alpha R\left[\left(U - \frac{c}{R}\right)(\varphi'' - \alpha^2\varphi) - U''\varphi\right] \qquad (1.1)$$

$$\left(R = \frac{U_0 h}{\nu}, \quad \lambda = i\alpha c, \quad c = c_r + ic_i, \quad U = x - x^3\right)$$

with the boundary conditions

$$\varphi = \varphi' = 0 \quad \text{for} \quad x = \pm 1 \tag{1.2}$$

We will solve the boundary value problem (1.1),(1.2) by the Galerkin method. Let us put

$$\varphi(x) = c_0\varphi_0^{(0)} + c_1\varphi_1^{(0)} + \ldots + c_N\varphi_N^{(0)} \tag{1.3}$$

Let us take the complete system of amplitudes of the normal perturbations in a fluid at rest as the system of basis functions $\varphi_n^{(0)}$; these functions and the corresponding decrements $\lambda_n^{(0)}$ are presented in [3] (*). The standard procedure of the method leads to a system of homogeneous algebraic equations for the coefficients of the expansion (1.3)

$$\sum_{n=0}^{N} c_n \{(\lambda_m - \lambda_n^{(0)})\delta_{mn} + i\alpha RH_{mn}\} = 0 \qquad (m = 0, 1, 2, \ldots, N) \tag{1.4}$$

Because of the oddness of the velocity profile of the fundamental flow, the matrix elements H_{mn} differ from zero only for subscripts of different evenness and for even n equal

$$H_{mn} = \frac{3}{I_n}[A_{mn}\,\alpha\tanh\alpha - B_{mn}\,\alpha\cot\alpha + C_{mn}] \tag{1.5}$$

$$I_n = \frac{\lambda_n^{(0)}}{4(\alpha^2 - \lambda_n^{(0)})}(\alpha^2 + \alpha\tanh\alpha - \alpha^2\tanh^2\alpha - \lambda_n^{(0)})$$

$$A_{mn} = -\frac{4\alpha^2}{\lambda_n^{(0)2}} - \frac{1}{4\alpha^2} + \frac{1}{\lambda_m^{(0)}} + \frac{4\alpha^2\lambda_n^{(0)} - 5\lambda_n^{(0)}\lambda_m^{(0)} + \lambda_m^{(0)2}}{(\lambda_n^{(0)} - \lambda_m^{(0)})^3}$$

$$B_{mn} = \frac{2\lambda_n^{(0)} - 4\alpha^2}{\lambda_n^{(0)2}} + \frac{1}{4\alpha^2} + \frac{1}{\lambda_m^{(0)}} + \frac{4\alpha^2\lambda_n^{(0)} - 3\lambda_n^{(0)}\lambda_m^{(0)} + \lambda_m^{(0)2} - 2\lambda_n^{(0)2}}{(\lambda_n^{(0)} - \lambda_m^{(0)})^3}$$

$$C_{mn} = \frac{\lambda_m^{(0)}(\lambda_m^{(0)} - 3\lambda_n^{(0)})}{3(\lambda_n^{(0)} - \lambda_m^{(0)})^2} + \frac{4\alpha^2}{\lambda_n^{(0)2}} + \frac{1}{4\alpha^2} + \frac{1}{\lambda_m^{(0)}} - \frac{\lambda_m^{(0)3} - 2\lambda_n^{(0)}\lambda_m^{(0)2} - 7\lambda_n^{(0)2}\lambda_m^{(0)} + 4\alpha^2\lambda_n^{(0)}(\lambda_n^{(0)} - \lambda_m^{(0)})}{(\lambda_n^{(0)} - \lambda_m^{(0)})^4}$$

$$(m = 1, 3, 5, \ldots;\ n = 0, 2, 4, \ldots)$$

For odd n the matrix elements are obtained from (1.5) by replacing all $\tanh\alpha$ and $\coth\alpha$ by $\coth\alpha$ and $\tan\alpha$, respectively.

For large values of the number R the solution of the boundary value problem (1.1) and (1.2) may be complex. Hence, to find the approximate solution (1.3) it is necessary to take a large number of basic functions. The maximum number of basic functions utilized in this computation is 18. This would permit finding the dependence of the 11 lower levels of the spectrum on αR with sufficient accuracy in the $0 < \alpha R < 1500$ range. The convergence of the method was estimated by a comparison of the results of approximations containing 14, 16, 17 and 18 basic functions for the wave number $\alpha = 1$. Approximations with 16 to 18 functions practically coincide in the range mentioned.

By a unitary transformation the matrix of coefficients of the system (1.4) may be reduced to real form. Then the problem of finding the decrements λ reduces to finding the eigenvalues of a real $(N + 1)$-th order matrix

$$(\lambda_n^{(0)}\delta_{mn} - (-1)^n\alpha RH_{mn})$$

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*) Petrov [4] first applied the basis utilized to the investigation of hydrodynamic stability problems.

for fixed α and R. The eigenvalues and eigenvectors of this matrix were found by an orthogonal powers method [5]. All the computations were performed on the "Aragats" electronic computer in Perm' University.

2. Let us consider the spectrum of the normal perturbation decrements.

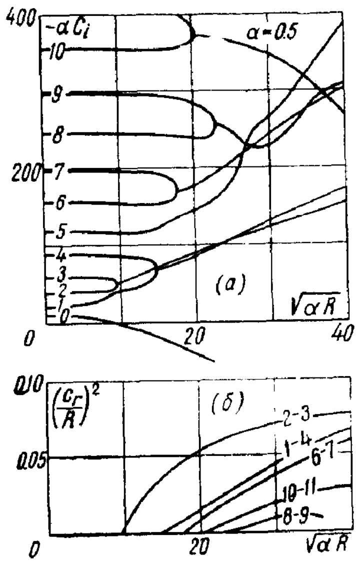

Fig. 1

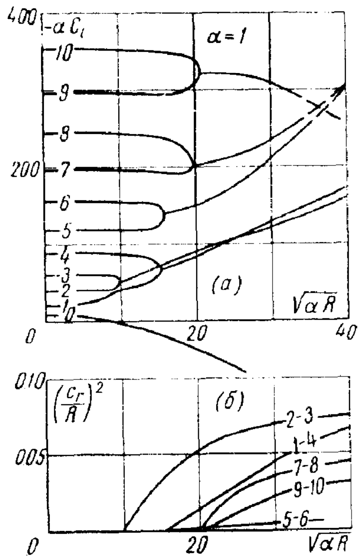

Fig. 2

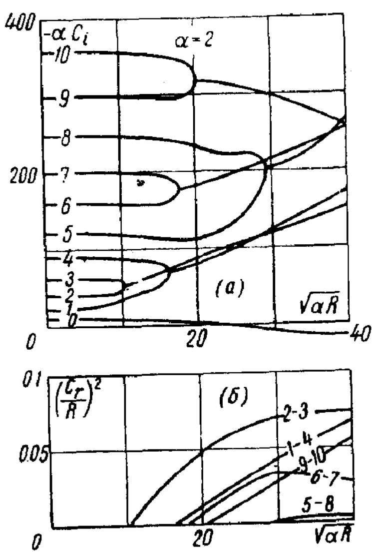

Fig. 3

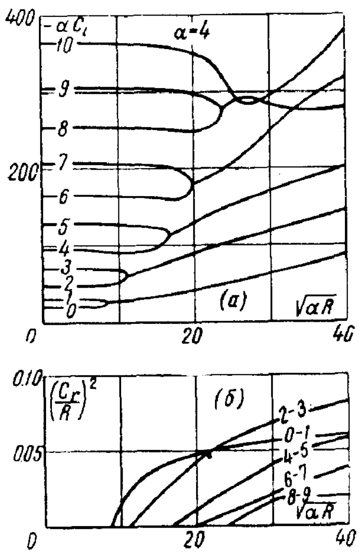

Fig. 4

The real part of the decrement $-\alpha c_i$ is pictured in Fig.1a as a function of the number $(\alpha R)^{1/2}$ for $\alpha = 0.5$. Fig.1b yields the dependence of the square of the perturbation phase velocity, measured in units of the fundamental flow velocity on $(\alpha R)^{1/2}$. Fig.2 refers to perturbations with wave number unity. The portions of the curves which are dashed in this figure correspond to that range of αR numbers where the 18-function approximation differes quantitatively from the 16-function approximation. Figs.3 and 4 yield the dependence of the decrements on the Reynolds number for $\alpha = 2, 4$, respectively.

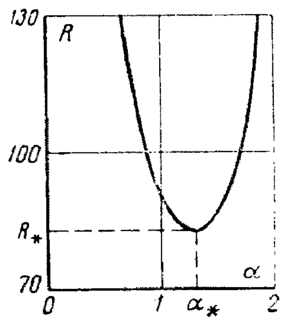

Fig. 5

In conformity with the general considerations [3], the perturbation decrements at low Reynolds numbers are real. As is seen from Figs.1 to 3, the lower decrement intersects the αR axis and changes sign. This means that, in contrast to Couette flow, a monotonous instability occurs in a flow with cubic profile. It originates at comparatively low Reynolds numbers. The neutral monotonous instability curve is pictured in Fig.5. The least critical value R_* yielding the beginning of the instability, is 83 and is reached for $\alpha_* = 1.3$. The mode of fluid motion given by this perturbation is of specific character and is discussed below; the given instability may be called the instability of the interface of opposite flows. The flow considered in [6] also possesses an analogous kind of instability. Let us note that the instability of the cubic profile with respect to monotonous perturbations is disclosed in an approximation with two functions [7] (*).

Table 1

x	$\varphi_r(x)$	$\varphi_i(x)$	x	$\varphi_r(x)$	$\varphi_i(x)$
—1.00	0.000	0.000	—0.50	4.166	—3.285
—0.95	0.104	—0.208	—0.45	4.482	—3.234
—0.90	0.396	—0.721	—0.40	4.748	—3.105
—0.85	0.827	—1.325	—0.35	4.963	—2.908
—0.80	1.339	—1.875	—0.30	5.130	—2.652
—0.75	1.881	—2.334	—0.25	5.254	—2.334
—0.70	2.418	—2.707	—0.20	5.337	—1.952
—0.65	2.923	—2.987	—0.15	5.387	—1.516
—0.60	3.385	—3.167	—0.10	5.413	—1.041
—0.55	3.800	—3.261	—0.05	5.425	—0.532
—0.50	4.166	—3.285	—0.00	5.429	0.000

As the Reynolds number increases in flows with an odd velocity profile, pairwise merger of the real decrements occurs with the formation of a complex conjugate pair. This means that two monotonous perturbations transform into two vibrational perturbations with the same damping velocity, travelling in opposite directions. The vibrational perturbations start to appear at αR numbers of about 100. Hence, for all the known levels of the perturbation spectrum of plane-parallel Couette flow [8], their real part starts to increase after the merger of the two real decrements, i.e. the stability relative to vibrational perturbations rises as the Reynolds number increases. There is a complex decrement in the flow under investigation whose real part diminishes as αR increases (the 10-11 level for $\alpha = 0.5$, and 9-10 for $\alpha = 1$ and 2). Extrapolation of the real part of this decrement to zero leads to the estimate $R_* \sim 10^4$. A value of the same order is obtained if one of the opposite streams of the flow with a cubic profile is compared with Poiseuille flow and the appropriate recalculation is made. To calculate the critical Reynolds number at which vibrational instability sets in (if it exists), it is understandably necessary to go forward into the domain of higher Reynolds numbers in the computation by taking a greater number of basis functions, or by using an asymptotic method.

*) Calculations with two functions yield $R_* = 69$, $\alpha_* = 1.6$. Extrapolation by means of square corrections to the decrements at $R = 0$ (see [3]) leads to the values $R_* \approx 50$, $\alpha_* \approx 1.5$

3. Let us now consider the perturbation mode of the investigated flow. The amplitude function $\varphi(x)$ of a monotonously increasing perturbation was calculated for developed motion with the parameters: $\alpha = 1$, $R = 1000$, $\lambda = -101.7$. The numerical values of the amplitudes are presented in Table 1 (the real part of the amplitude $\varphi_r(x)$ is an even, and the imaginary part $\varphi_i(x)$ an odd function of x). To investigate the mode of the motion in the perturbations, a perturbation streamline may be constructed

$$\mathrm{Re}\,\Psi = [\varphi_r \cos(\alpha z - \lambda_i t) - \varphi_i \sin(\alpha z - \lambda_i t)]\, e^{-\lambda_r t} = \mathrm{const}\, e^{-\lambda_r t}$$

The perturbation has the form of fixed ($\lambda_i = 0$) cells with motion that is symmetric relative to the center of the cell (Fig.6). As is seen from the figure, the perturbation penetrates identically into both halves of the fundamental flow. The addition of this perturbation to the fundamental flow makes the wavy interface between the two opposing flows, i.e. the violation of stability is associated with the instability of the plane interface of the flows relative to small perturbations.

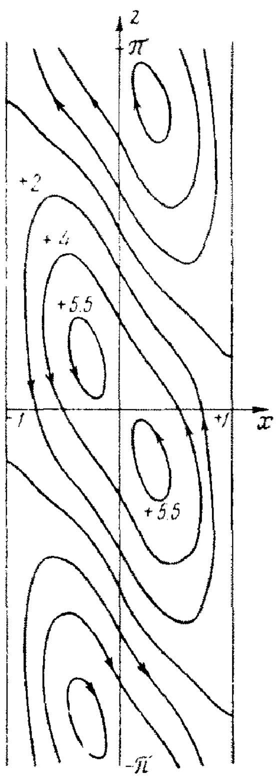

Fig. 6

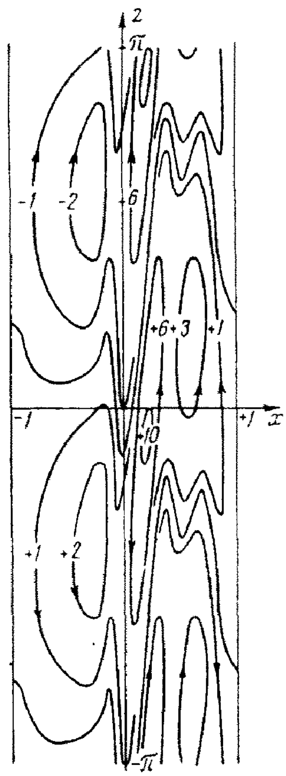

Fig. 7

The vibrational perturbations are of essentially different nature. For example, let us consider the perturbation corresponding to $\lambda_{9,10}$ with the wave number $\alpha = 1$ for $\alpha R = 1000$ and $c_r > 0$. Numerical values of the amplitude function for this perturbation are presented in Table 2. The perturbation moves along the z-axis with velocity 0.159 units of the fundamental flow velocity, and its amplitude diminishes with time. The velocity distribution in the perturbation is seen in Fig.7, in which the perturbation streamlines at $t = 0$ are pictured. The perturbation is almost completely localized in the domain of positive x, i.e. in that part of the flow where the direction of propagation of the perturbation agrees with the direction of the velocity of the fundamental motion. The magnitude of the perturbation velocity in this part of the flow is an order of magnitude greater than the perturbation velocity in the opposing flow. The perturbation with complex-conjugate decrement (negative phase velocity) is localized in the $-1 < x < 0$ domain. The amplitude of this perturbation $\bar{\varphi}(x)$ is obtained from Table 2 by replacing $\varphi_r(x)$ by $\varphi_i(-x)$ as well as $\varphi_i(x)$ by $\bar{\varphi}_r(-x)$.

Table 2

x	$\varphi_r(x)$	$\varphi_i(x)$	x	$\varphi_r(x)$	$\varphi_i(x)$
−1.00	0.000	0.000	0.20	9.732	2.897
−0.90	0.191	0.303	0.22	9.974	1.111
−0.80	0.254	0.877	0.25	9.455	−1.416
−0.70	0.248	1.308	0.30	6.936	−4.151
−0.60	0.296	1.654	0.35	4.071	−4.432
−0.50	0.400	1.914	0.40	2.444	−3.110
−0.40	0.473	2.089	0.45	2.334	−1.714
−0.30	0.637	2.379	0.50	2.958	−1.198
−0.25	0.915	2.480	0.55	3.415	−1.538
−0.20	1.258	2.290	0.60	3.366	−2.143
−0.15	1.311	1.729	0.65	3.017	−2.426
−0.10	0.660	1.224	0.70	2.665	−2.104
−0.05	−0.667	1.634	0.80	1.955	−0.283
0.00	−1.319	3.487	0.85	1.361	0.355
0.05	−0.224	6.100	0.90	0.699	0.433
0.10	3.069	7.655	0.95	0.189	0.166
0.15	7.152	6.556	1.00	0.000	0.000

As is known, the nonzero phase velocity of the neutral perturbation makes isolated points at which the velocity of the fundamental motion equals the velocity of the perturbation. A perturbation separating the streams is localized around these points. The generation of one such layer (a sharp peak in the stream function around the points $x = 0.2$, $z = -0.2$) can be seen in Fig.7. This makes plausible the assumption expressed above on the existence of a vibrational instability of the same nature as a Poiseuille flow instability in the flow under consideration. A final deduction of the presence of this instability may be made after an additional investigation.

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